SECONDARY FLOW ABOUT A MAGNETIZED SPHERE ROTATING IN VISCOUS CONDUCTING FLUID

BY
SUNIL DATTA

ABSTRACT
The problem of secondary motion induced by the steady rotation of a magnetized sphere in an infinite incompressible viscous conducting fluid is considered. It is found that the secondary flow adds nothing to the couple required tomaintain the motion and the effect of the magnetic field is to damp the secondary velocity field.

Introduction. The steady rotation of a sphere, magnetized along the axis of rotation, in an infinite incompressible viscous conducting fluid was considered by the author [1] under the assumption that the fluid moves in concentric circles whose centres lie on the axis of rotation. Actually since the centrifugal force is greatest in the neighbourhood of the equator of the sphere, the fluid particles will recede from the sphere at the equator and approach it again at the poles. Thus combined with the motion about the axis of rotation, there will be a circulatory motion in planes containing the axis of rotation. This secondary flow for the case of an incompressible viscous fluid has been investigated by several authors [2–4]. In the present paper the analysis has been extended to the case of a magnetized, sphere.

Basic Equations. With the usual notation the basic equations of magneto-hydrodynamics in the non-dimensional form are

(1 a, b, c, d) $$\operatorname{curl}\bar{E} = 0,\ \bar{J} = R_m \operatorname{curl}\bar{B},\ \bar{J} = (\bar{E} + \bar{V}\times\bar{B}),\qquad \operatorname{div}\bar{B} = 0,$$

(2 a, b) $$R(\bar{V}\cdot\nabla)\bar{V} = -\nabla P + \nabla^2\bar{V} + M(\bar{J}\times\bar{B}),\qquad \operatorname{div}\bar{V} = 0,$$

where $R(=(a^2\Omega/\nu))$ is the Reynolds number, $R_m(=4\pi\sigma a^2\Omega\mu)$ is the magnetic Reynolds number and $M(=(\sigma/\rho\nu)a^2B_m^2)$ is the square of Hartmann number.

From (1) we get

(3) $$\bar{E} = -\operatorname{grad}\phi \text{ and } \nabla^2\phi = \operatorname{div}(\bar{V}\times\bar{B}).$$

To solve the problem we make use of the following perturbation expansions

(4 a, b, c) $$\begin{cases} \bar{V} = \bar{V}_0 + R\bar{V}_1 + M\bar{V}_1' + RM\bar{V}_2 + \cdots, \\ P = P_0 + RP_1 + MP_1' + RMP_2 + \cdots, \\ \phi = \phi_0 + R\phi_1 + \cdots. \end{cases}$$

Received Feb. 7, 1965.

The magnetic Reynolds number is assumed to be small so that the magnetic field, $\bar{B}[= -\nabla(\frac{1}{2}(z/r^3))]$, of the sphere, remains unaffected by the velocity field. Inserting this value of $\bar{B}$ and the expansions of $\bar{V}$ and ϕ in (3) we get

$$\text{(5 a, b)} \qquad \begin{cases} \nabla^2\phi_0 = \operatorname{div}(\bar{V}_0 \times \bar{B}), \\ \text{and} \\ \nabla^2\phi_1 = \operatorname{div}(\bar{V}_1 \times \bar{B}). \end{cases}$$

Again using the perturbation expansions (4) in (2a) we get the following equation

$$\text{(6 a, b, c, d)} \qquad \begin{cases} 0 = -\nabla P_0 + \nabla^2 \bar{V}_0, \\ (\bar{V}_0 \cdot \nabla)\, \bar{V}_0 = -\nabla P_1 + \nabla^2 \bar{V}_1, \\ 0 = -\nabla P_1' + \nabla^2 \bar{V}_1' + (\bar{E}_0 + \bar{V}_0 \times \bar{B}) \times \bar{B}, \\ (\bar{V}_0 \cdot \nabla)\bar{V}_1 + (\bar{V}_1' \cdot \nabla)\bar{V}_0' = -\nabla P_2 + \nabla^2 \bar{V}_2 + \\ \qquad + (\bar{E}_1 + \bar{V}_1 \times \bar{B}) \times \bar{B}. \end{cases}$$

The problem is to be solved subject to the following boundary conditions

(i) No-slip condition at the surface of the sphere

(ii) Continuity of normal component of current density vector and continuity of tangential component of electric intensity vector at the surface of the sphere.

(iii) Vanishing of the quantities at infinity.

Solution. Cylindrical polar coordinates $(\tilde{\omega}, \theta, z)$ with velocity components (u, v, w) will be used in writing down the soutions. The equation (6a) with $V_0 = \tilde{\omega}$ at $r(=\sqrt{\tilde{\omega}^2 + z^2}) = 1$ yields [5]

$$\text{(7)} \qquad \bar{V}_0 = \left(0, \frac{\tilde{\omega}}{r^3}, 0\right).$$

Again the solution of (6b) can be written down [3] in terms of a stream function

$$\text{(8)} \qquad \psi_1 = \frac{\tilde{\omega}^2 z}{8} \frac{(r-1)^2}{r^5}, \quad \bar{V}_1 = \left(\frac{1}{\tilde{\omega}} \frac{\partial \psi_1}{\partial z}, 0, -\frac{1}{\tilde{\omega}} \frac{\partial \psi_1}{\partial \tilde{\omega}}\right).$$

When $\bar{V}_1$ is inserted in (5b) we get

$$\nabla^2 \phi_1 = 0,$$

which together with the boundary condition gives $\bar{E}_1 = 0$.

The solution of equation (6c) has been obtained elsewhere [1] and is

$$\bar{V}_1' = (0, \tilde{\omega} g_1, 0),$$

where

$$g_1 = \frac{1}{24}\left(\frac{z^2}{r^9} + \frac{1}{7r^7}\right) + \frac{3\lambda}{16}\left(\frac{z^2}{r^8} + \frac{1}{3r^6}\right) + \frac{E}{r^3} + F\left(\frac{5z^2}{r^7} - \frac{1}{r^5}\right),$$

with

$$\lambda = -\frac{1}{2}\left(\frac{1+\sigma_r}{3+2\sigma_r}\right), \; E = \frac{1+3\sigma_r}{140(3+2\sigma_r)}, \; F = -\frac{3-2\sigma_r}{480(3+2\sigma_r)},$$

σ_r being the ratio of the conductivities of the sphere and the fluid medium.

Using the above results, equation (6d) in cylindrical polar-coordinates wit axial symmetry can be written as

(9 a, b, c)

$$\begin{cases} -\dfrac{2\tilde{\omega}g_1}{r^3} = -\dfrac{\partial p_2}{\partial \tilde{\omega}} + \nabla^2 u_2 - \dfrac{u_2}{\tilde{\omega}^2} - \dfrac{\tilde{\omega}}{16}\left(\dfrac{3z^2}{r^5} - \dfrac{1}{r^3}\right) \\ \quad \times \left[\dfrac{3z^2}{r^5}\left(\dfrac{1}{r^3} - \dfrac{2}{r^4} + \dfrac{1}{r^5}\right) - \dfrac{3}{2}\dfrac{z^2\tilde{\omega}^2}{r^5}\left(\dfrac{3}{r^5} - \dfrac{8}{r^6} + \dfrac{5}{r^7}\right)\right. \\ \qquad\qquad + \dfrac{1}{2}\left(\dfrac{3z^2}{r^5} - \dfrac{1}{r^3}\right) \\ \quad \left. \times \left\{\dfrac{1}{r^3} - \dfrac{2}{r^4} + \dfrac{1}{r^5} - z^2\left(\dfrac{3}{r^5} - \dfrac{8}{r^6} + \dfrac{5}{r^7}\right)\right\}\right] \\ 0 = \nabla^2 v_2 - \dfrac{v_2}{\tilde{\omega}^2} \\ 0 = -\dfrac{\partial p_2}{\partial z} + \nabla^2 w_2 + \dfrac{3}{16}\dfrac{\tilde{\omega}^2 z}{r^5}\left[\dfrac{3z^2}{r^5}\left(\dfrac{1}{r^3} - \dfrac{2}{r^4} + \dfrac{1}{r^5}\right)\right. \\ \quad - \dfrac{3}{2}\dfrac{z^2\tilde{\omega}^2}{r^5}\left(\dfrac{3}{r^5} - \dfrac{8}{r^6} + \dfrac{5}{r^7}\right) + \dfrac{1}{2}\left(\dfrac{3z^2}{r^5} - \dfrac{1}{r^3}\right) \\ \quad \left. \times \left\{\dfrac{1}{r^3} - \dfrac{2}{r^4} + \dfrac{1}{r^5} - z^2\left(\dfrac{3}{r^5} - \dfrac{8}{r^6} + \dfrac{5}{r^7}\right)\right\}\right] \end{cases}$$

The boundary conditions are that, on $r = 1$,

$$u_2 = v_2 = w_2 = 0,$$

and each of the functions tends to zero as r tends to infinity. The equation for v_2 is satisfied by taking $v_2 = 0$ throughout the fluid.

To solve the equations for u_2 and w_2, let

$$u_2 = \frac{1}{\tilde{\omega}}\frac{\partial \psi_2}{\partial z}, \; w_2 = -\frac{1}{\tilde{\omega}}\frac{\partial \psi_2}{\partial \tilde{\omega}}$$

Substituting these values the equation for ψ_2 is obtained from (9 a, c) to be

(10) $$\Delta^4 \psi_2 = \tilde{\omega}^2[z f_1(r) + z^3 f_2(r)],$$

where

$$\Delta^2 = \frac{\partial^2}{\partial \tilde{\omega}^2} - \frac{1}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} + \frac{\partial^2}{\partial z^2},$$

$$f_1 = \frac{1}{16}\left(\frac{1600F}{r^{12}} + \frac{66\lambda - 15}{r^{13}} + \frac{28}{r^{12}} + \frac{7}{r^{15}}\right),$$

$$f_2 = \frac{12E}{r^8} - \frac{36F}{r^{10}} + \frac{6\lambda + 3}{16r^{11}} - \frac{17}{21r^{12}} + \frac{9}{16r^{13}},$$

and $\psi_2 = (\partial\psi_2/\partial r) = 0$, on $r = 1$ and $\psi_2 \to 0$ as $r \to \infty$.
Writing $\psi_2 = \tilde{\omega}^2[zF_1(r) + z^3F_2(r)]$ we get ordinary linear differential equations for the functions $F_1(r)$ and $F_2(r)$. These are

$$D(D-2)(D+7)(D+9)F_2 = \frac{1}{16}\left(\frac{1600F}{r^8} + \frac{66\lambda - 15}{r^9} + \frac{28}{r^{10}} + \frac{7}{r^{11}}\right),$$

and

$$D(D-2)(D+3)(D+5)F_1 = \frac{12E}{r^4} - \frac{36F}{r^6} + \frac{6\lambda + 3}{11r^7} - \frac{17}{21r^8} + \frac{9}{16r^9}$$
$$- (12r^4F_2'' + 96r^3F_2'),$$

where $D = (1/r)(d/dr)$.

The equations have been solved to give the functions F_1 and F_2 which satisfy the boundary conditions

$$F_1 = F_2 = F_1' = F_2' = 0, \text{ on } r = 1,$$

and tend to zero as r tends to infinity. The functions F_1 and F_2 for $\sigma_r = 1$ are given below

$$F_1(r) = \left[-\frac{976}{r^3} - \frac{2857}{r^4} + \frac{7435}{r^5} - \frac{243}{r^6} - \frac{3778}{r^7}\ln r \right.$$
$$\left. - \frac{1378}{r^7} - \frac{2133}{r^8} + \frac{152}{r^9}\right] \times 10^{-6}, \tag{11}$$

and

$$F_2(r) = \left[-\frac{6062}{r^7} + \frac{521}{r^8} + \frac{298}{r^9} + \frac{8900}{r^9}\ln r + \frac{4861}{r^{10}} + \frac{382}{r^{11}}\right] \times 10^{-6} \tag{12}$$

The stream function for the secondary flow is given by

$$\psi = \psi_1 + M\psi_2 = \tilde{\omega}^2\left[\frac{z}{8}\frac{(r-1)^2}{r^5} + M\left\{zF_1(r) + z^3F_2(r)\right\}\right] \tag{13}$$

It is easy to see that the secondary velocity field, as obtained above, contributes nothing to the couple required to maintain the motion.

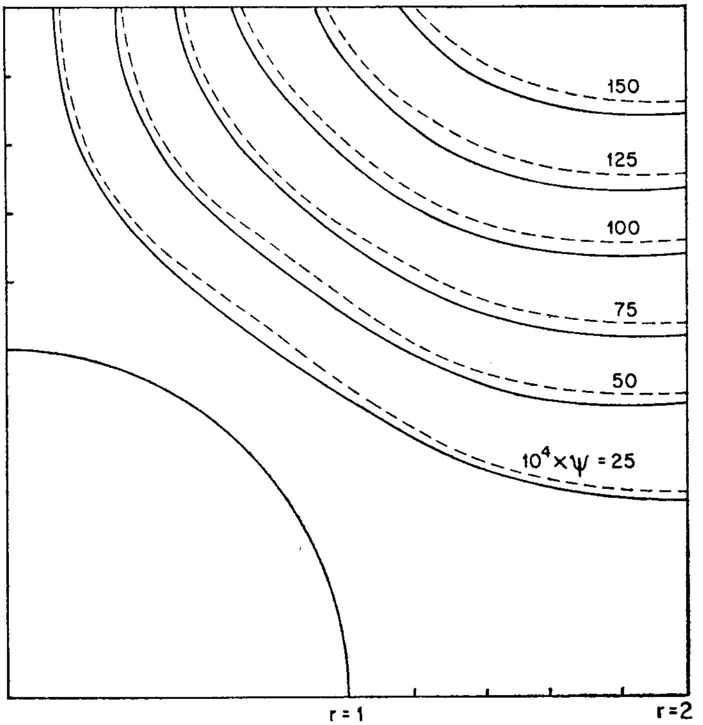

Fig. Stream line pattern for the secondary flow. — M = 0, - - - M = 1

Stream line pattern for the secondary flow, in the plane containing the axis, is given in the figure for $M = 0$ and $M = 1$. The graph suggests that the secondary velocity field decreases on account of the magnetic field.

Acknowledgement. I am grateful to Prof. Ram Ballabh for his help and guidance in the preparation of this paper.

REFERENCES

1. SUNIL DATTA, *J. Phys. Soc. Japan*, **19** (1964), 392.
2. W.G. BICKLEY, *Phil. Mag.* **25** (1938), 746.
3. W.D. COLLINS, *Mathematika* **2** (1955), 42.
4. W.L. HABERMAN, *Phys. of Fluids*, **5** (1962), 625.
5. H. LAMB, *Hydrodynamics*, p. 588–9.

DEPARTMENT OF MATHEMATICS AND ASTRONOMY,
LUCKNOW UNIVERSITY, LUCKNOW,
INDIA